Combinatorics of Coxeter Groups Seminar Outline

Fall 2022

- If a proof uses material from a different lecture than yours, please remind the audience what the corresponding statement is by writing it out on the blackboard.
- If you don't understand notation(we use a lot of references) or have any questions, please email me ccl2166@columbia.edu even if we have already met before your talk.
- If you find a proof hard to understand or find it annoying, you can instead give examples demonstrating the theorem or lemma instead of presenting the proof.
- Material in parenthesis are optional and should only be covered if there's time. Also "until Theorem A" means that you end by covering Theorem A.

1 Fundamentals

(1) **Definitions and Examples**

(a) Cover [7, Chapter 1.1.1] and [7, Chapter 1.1.2] especially the part about strand diagrams.

(b) Cover [2, Example 1.2.7] and [2, Example 1.2.8]. If the definition of a regular polytope is confusing, note that in dimension 3, a polytope is regular if all the faces are the same and around each vertex, the way the faces meet are the same. The definition in the book is the generalization of this to higher dimensions.

(c) (Cover [7, Chapter 1.1.3].)

(d) (Cover [2, Example 1.2.9].)

(2) **Finite Coxeter Groups and Root Systems**

(a) Cover [7, Chapter 1.1.6-1.1.8].

(b) Cover [13, Chapter 1.1, pg 3]. Define root systems following [13, Chapter 1.2] starting at "For flexibility in some future arguments..." Cover [13, Chapter 2.9] pg 29 first 3 paragraphs.

(c) Go over the root systems A_1 and the rank 2 root systems A_2, B_2, C_2, G_2 following [17, Section 2].

(3) **Lengths, Descents and Exchange Conditions**

(a) Cover [7, Chapter 1.2.1], especially Example 1.39. In particular, prove that if two strands every cross each other in the strand diagram, then they can always be removed/straightened out (Hint: Use induction). Prove [2, Proposition 1.5.2]. Leave [7, Example 1.40] for the end of the talk and only present if strand diagrams for type B were introduced in previous talks.

(b) Cover [7, Chapter 1.2.2-1.2.3] Please use the notation $D_R(w)$ for $\mathcal{R}(w)$ and $D_L(w)$ for $\mathcal{L}(w)$.

(c) (State [2, Chapter 1.5.1] and prove the direction $(iii) \implies (ii)$.)

(4) **Reflections, the Longest element and Matsumoto's theorem**

(a) Define reflections and the elements t_i and prove Lemma 1.3.1 [2, page 12]. Define T_L and T_R and prove [2, Corollary 1.4.5]

(b) Cover [2, Chapter 2.3] up to the example of Proposition 2.3.4 in S_5. (Just state [2, Proposition 2.3.1]).

(c) Do examples for the dihedral group. (Talk to Cailan)

(d) Cover [7, Example 1.52] and [7, Theorem 1.56].

(e) (State [2, Chapter 1.5.1] and prove the direction $(iii) \implies (ii)$.)

2 Combinatorics of Posets

(5) **Principle of Inclusion Exclusion**

(a) Cover [18] up to Section 3.3.

(b) (Cover [8, Sections 2, 3].)

(6) **Posets and Lattices**

(a) Cover [19, Chapter 5.1] (only cover C_n, B_n, D_n in the examples) ending on page 143 at the sentence "... this diagram is exactly like the one for B_3 in Figure 5.1." Start again on page 144 at the sentence "To define isomorphism, ..." until the end of the page. Prove Proposition 5.1.3(f).

(b) Cover the direct product of posets, [19, Chapter 5.2] starting on page 146 at "Our third method ...." End on page 147 on "But this chain is not maximal...

(c) Cover [19, Chapter 5.3] up until Proposition 5.3.2 (Only cover $(a) - (f)$)

(7) **The Möbius Function**

(a) Cover [19, Chapter 5.4] up to Proposition 5.4.5, skipping the proof of Theorem 5.4.3.

(b) Follow [19, Chapter 5.5] up until Equation 5.13. Then start again on page 161 until the end of 162.

(c) State [19, Theorem 5.5.7] and [19, Theorem 5.5.8] and go over their proofs if there is time.

(d) Cover [3, Chapter 10] starting at "The Mobius function is of great importance" and ending at the Riemann Hypothesis.

3 Combinatorics of the Bruhat Order

(8) **The Bruhat Order (of S_n)**

(a) Cover [2, Chapter 2.1], skipping Example 2.1.3 if not familiar.

(b) Do many examples illustrating Theorem 2.1.5 and only prove it if there's time.

(9) **Properties of the Bruhat Order**

(a) Cover [2, Chapter 2.2] skipping Corollary 2.2.8 and Lemma 2.2.10. Demonstrate the Theorems and Proposition using the Bruhat order of the dihedral group.

(b) Show that W has a unique element at the top of the Bruhat order using Proposition 2.2.9

(10) **Parabolic Subgroups and the Tableau Criterion**

(a) Cover [2, Chapter 2.4] skipping the proof of Proposition 2.4.1 while skipping Corollary 2.4.5 and Corollary 2.4.6 entirely. Starting again at "Let us exemplify the preceeding in the case of symmetric groups" on page 41.

(b) Cover [2, Chapter 2.5] up to Proposition 2.5.1. Follow [2, Chapter 2.6] starting on page 47.

4 Enumeration and generating functions

(11) **Generating Functions**

(a) Follow [9, Section 1] skipping Example 1.

(b) Follow [9, Section 2] starting at "We have just seen addition ..." and ending at the proof of Theorem 1.

(c) Follow [9, Section 4] and [9, Section 5].

(12) $q-$**analogs**

(a) Define $q-$numbers (Equation (3.2) on page 75) and $q-$analogues and then prove that the generating function for inversions is a $q-$analog for $n!$ [19, Theorem 3.2.1].

(b) Start on page 77 of [19] and go until Theorem 3.2.4.

(c) Cover [5, Section 6.4].

(13) **Poincare Series**

(a) Cover [2, Chapter 7.1] skipping Corollary 7.1.8 and the paragraph above it and ending at Equation (7.9).

(b) (State [2, Theorem 7.1.10].)

(14) **Eulerian Polynomials**

(a) Cover [19, Theorem 4.24, Theorem 4.25] but replace q with t and multiply both sides by $(1-t)^{n+1}\dfrac{x^n}{n!}$ to instead obtain

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{(1-t)e^{(1-t)x}}{1-te^{(1-t)x}}$$

(b) Cover [2, Chapter 7.2] up until Equation (7.12).

(c) (Prove Euler's formula for the values of the alternating Riemann zeta function following [12] up to Equation (5).)

5 Combinatorics of Young Tableaux

(15) **(Semi)Standard Young Tableaux**

(a) Follow [22, Section 4.2] up to definition 4. f^λ denotes the number of standard $\lambda-$tableau. State the hook-length formula [22, Theorem 25].

(b) $\delta_n = (n, n-1, \ldots, 2, 1)$ is called a staircase partition. State [2, Theorem 7.4.7] and then prove [2, Corollary 7.4.8].

(c) State the Hook-Content formula and do some examples. Then state the corresponding $q-$analog and do some examples [16, Sections 2, 3].

(d) (Compute an explicit formula for the number of Standard Young Tableaux for the partition (n, n).)

(16) **Catalan Numbers and Fully Commutative Elements**

(a) Follow [15, Section 1] up to Theorem 1 (only cover parenthesis and dyck paths unless you have extra time). Cover [15, Section 2].

(b) Follow [15, Section 3] ending before Section 3.1. Prove that there is a bijection between 321 avoiding permutations of S_n and Dyck paths of length $2n$ following Brian M. Scott's answer in [21].

(c) Prove that 321 avodiing permutations of S_n are the same as fully commutative elements (C_1-equivalent) of S_n following [1, Theorem 2.1 (a), (b)].

(17) **Young's Lattice and Differential Posets**

(a) Follow [20, Chapter 5.1].

6 Extra Topics

(18) **Log Concavity, Unimodality and Matroids**

(a) Follow [4, Section 1] up to Theorem 1.2. Recall the definition of descents and the Eulerian polynomial (the first paragraph of[4, Section 8.2]) and state Theorem 8.14. Write down $P_1(t)$ to $P_6(t)$ following [12].

(b) Read through [10, Chapter 1.1]. Cover [10, Chapter 1.2] starting at Definition 1.3 and ending at Example 1.6.

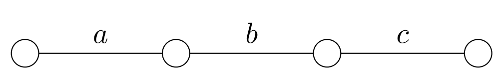
(c) Cover [10, Definition 2.32] and [10, Example 2.33].

(d) (Follow [10, Chapter 2.42] starting at Definition 2.38 and ending at Definition 2.41, skipping the modular part. State [10, Theorem 2.45].)

(19) **Matroids and the Work of June Huh**

(a) Cover the chromatic polynomial portions of [11], [6].

(b) Cover [10, Section 1.4.2] up until Theorem 4.1. Compute the matroid corresponding to the graph



(1)

You should end up with $B_n = U_{n,n}$. Follow page 24 from "The first important question..." to "are called graphic." State that $U_{2,4}$ is not graphic and that all graphic matroids are representable.

(c) State [10, Definition 2.12] and compute the ranks for $B_n = U_{n,n}$ the boolean algebra and $U_{2,4}$. State [10, Definition 2.16] and compute the flats for $B_n = U_{n,n}$ the boolean algebra and $U_{2,4}$. Then state the following

Theorem 6.1. *Given a matroid M, let $(\mathscr{F}(M), \subseteq)$ be the poset where $\mathscr{F}(M)$ is the set of all flats of M and $\subseteq$ is set inclusion. Given $(L, \leq)$ a lattice, TFAE*

(1) $(L, \leq)$ is a geometric lattice (atomic and rank function is semimodular).

(2) $(L, \leq)$ is isomorphic as a poset to $(\mathscr{F}(M), \subseteq)$ for some matroid M.

(d) Define the characteristic polynomial of a graded poset [19, Chapter 5.6, Equation 5.15] and cover Proposition 5.6.1 $(a), (b), (c)$. Now define the characteristic polynomial $\chi(M, t)$ of a matroid M to be $\chi(\mathscr{F}(M), t)$ the characteristic polynomial of the corresponding geometric lattice.

(e) State the following theorem

Theorem 6.2. *Given a graph G with chromatic polynomial $C(G, t)$ and let G_M be the corresponding matroid. Then*

$$C(G, t) = t^c \chi(G_M, t)$$

where c is the number of connected components of G.

and show this is true when $G =$ graph in Eq. (1).

(f) Follow [14] page 2, the 2nd paragraph.

(g) (Summarize Huh's life story [11], [6].)

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